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Rational Approximation, III

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Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function. Denote $M(r) = \max_{|z|=r} |f(z)|$; $S_n(z)$ denotes the nth partial sum of $f(z)$. As usual, the order ρ $(0 \leqslant \rho \leqslant \infty)$ of $f(z)$ is

$$\limsup_{r\to\infty} \frac{\log \log M(r)}{\log r}.$$

If $0 < \rho < \infty$, then the type τ and the lower type ω $(0 < \omega \leqslant \tau < \infty)$ of $f(z)$ are

$$\begin{matrix}\tau \\ \omega\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(r)}{r^{\rho}}.$$

Recently approximation to e^{-x} on $[0, \infty)$ has attracted the attention of several mathematicians. In [3], it has been established that $e^{-|x|}$ can be approximated on $(-\infty, \infty)$ by reciprocals of polynomials of degree n with an error $\leqslant C_1(\log n)\, n^{-1}$, but not better than $C_2 n^{-1}$. Further, we have shown that $e^{-|x|}$ can be approximated on $(-\infty, \infty)$ by rational functions of degree n with an error $\leqslant e^{-C_3(n)^{1/2}}$ but not better than $e^{-C_4(n)^{1/2}}$. In this note we obtain error bounds to $|x|\, e^{-|x|}$ on $(-\infty, \infty)$ by reciprocals of polynomials of degree n and also by rational functions of degree n. We show here that the minimum error by rational functions of degree n is much smaller than the one obtained by reciprocals of polynomials of degree n. Throughout our work $C_1, C_2, C_3, \ldots$ denote suitable positive constants, and ϵ, $0 < \epsilon < 1$, is arbitrary.

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LEMMAS

LEMMA 1 [5, p. 11]. *There exists a sequence of rational functions* $\{Q_n(x)\}_{n=1}^{\infty}$ *for which, for all* $n \geq 5$,

$$\| |x| - Q_n(x) \|_{L_\infty[-1,1]} \leq 3e^{-n^{1/2}}.$$

In fact, one can take

$$Q_{n+1}(x) = x \left[\frac{P_n(x) - P_n(-x)}{P_n(x) + P_n(-x)} \right],$$

where

$$P_n(x) = \prod_{i=0}^{n-1} (x + \xi^i), \qquad \xi = \exp(-1/n^{1/2}).$$

Remark. For every positive A,

$$\| |x| - AQ_{n+1}(x/A) \|_{L_\infty[-A,A]} \leq 3Ae^{-n^{1/2}}.$$

This follows easily from Lemma 1.

LEMMA 2 [6, p. 232]. *There is a polynomial* $P_n(x)$ $(n = 1, 2,...)$ *of degree* $\leq 2n$ *such that*

$$\| |x| - (1/P(x)) \|_{L_\infty[-1,1]} \leq \pi^2/2n.$$

Remark 1 [6, p. 234].

$$P_n(x)^{-1} \leq |x| \quad \text{for} \quad |x| \geq 1.$$

Remark 2. For each $A > 0$,

$$\left\| |x| - \frac{A}{P_n(x/A)} \right\|_{L_\infty[-A,A]} \leq \frac{A\pi^2}{2n}.$$

This follows easily from Lemma 2.

LEMMA 3 [8, p. 68]. *Let* $P(x)$ *be a polynomial of degree at most* n *satisfying* $|P(x)| \leq M$ *on* $[a, b]$. *Then outside* $[a, b]$,

$$|P(x)| \leq M \left| T_n\left(\frac{2x - b - a}{b - a}\right) \right|.$$

LEMMA 4 [3, p. 22]. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geq 0$ $(k \geq 1)$, *be an entire function of order* ρ $(0 < \rho < \infty)$, *type* τ, *and lower type* ω

$(0 < \omega \leqslant \tau < \infty)$. *Then there exists a constant* C_5 (>0) *and a sequence of polynomials* $\{P_n(x)\}_{n=1}^{\infty}$ *of degree n such that, for* $n > 1$,

$$\left\| \frac{1}{f(|x|)} - \frac{1}{P_n(x)} \right\|_{L_\infty(-\infty,\infty)} \leqslant \frac{C_5(\log n)^{1/\rho}}{n}.$$

LEMMA 5 [3, p. 122]. *Let* $f(z)$ *satisfy the assumptions of Lemma* 4. *Then there exists a constant* C_6 (>0) *and a sequence of rational functions* $\{r_n(x)\}_{n=1}^{\infty}$ *of degree n such that, for any* $n \geqslant 1$,

$$\|(1/f(|x|)) - r_n(x)\|_{L_\infty(-\infty,\infty)} \leqslant e^{-C_6 n^{1/2}}.$$

LEMMA 6 [7]. *Under the same assumptions, we have for the polynomials* $P_n(x) = \sum_{k=0}^{n} a_k x^k$,

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} < 1.$$

THEOREMS

THEOREM 1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, *be an entire function of order* ρ $(0 < \rho < \infty)$, *type* τ, *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Then there exists a polynomial* $P_n{}^*(x)$ *of degree n for which, for all* $n > 1$,

$$\left\| \frac{|x|}{f(|x|)} - \frac{1}{P_n{}^*(x)} \right\|_{L_\infty(-\infty,\infty)} \leqslant \frac{C_9(\log n)^{2/\rho}}{n}. \tag{1.1}$$

Remark. If $f(z)$ is even, then $2/\rho$ in (1.1) can be replaced by $1/\rho$.

Proof. By Remark 2 following Lemma 2, and by Lemma 4, there exist polynomials $P(x)$ and $q(x)$ for which

$$\| |x| - (1/P(x))\|_{L_\infty[-A,A]} \leqslant A\pi^2/2n, \tag{1.2}$$

$$\left\| \frac{1}{f(|x|)} - \frac{1}{q(x)} \right\|_{L_\infty[-A,A]} \leqslant \frac{C_8(\log n)^{1/\rho}}{n}. \tag{1.3}$$

To obtain bounds for $x \in (-\infty, \infty)$, we note that

$$\left| \frac{|x|}{f(|x|)} - \frac{1}{P(x)\, q(x)} \right|$$
$$\leqslant \frac{1}{f(|x|)} \left| |x| - \frac{1}{P(x)} \right| + \frac{1}{P(x)} \left| \frac{1}{f(|x|)} - \frac{1}{q(x)} \right|. \tag{1.4}$$

For $0 \le |x| \le (4\omega^{-1} \log n)^{1/\rho}$,

$$\frac{1}{f(|x|)}\left| |x| - \frac{1}{P(x)} \right| \le C_9 (\log n)^{1/\rho} n^{-1}. \tag{1.5}$$

For $|x| > (4\omega^{-1} \log n)^{1/\rho}$, by using the definition of lower type and the fact that

$$P(x)^{-1} \le |x| \quad \text{for} \quad |x| > (4\omega^{-1} \log n)^{1/\rho},$$

we get, for all large n,

$$\frac{1}{f(|x|)}\left| |x| - \frac{1}{P(x)} \right| \le \frac{2|x|}{f(|x|)} \le \frac{2|x|}{e^{(\omega/2)|x|^{\rho}}} \le n^{-3}. \tag{1.6}$$

Similarly we get, for $0 \le |x| \le (4\omega^{-1} \log n)^{1/\rho}$, by using Remark 2 following Lemma 2 with $A = (4\omega^{-1} \log n)^{1/\rho}$, and Lemma 4,

$$\begin{aligned} \frac{1}{P(x)}\left| \frac{1}{f(|x|)} - \frac{1}{q(x)} \right| &\le \left(|x| + \frac{C_{10}(\log n)^{1/\rho}}{n} \right)\left(\frac{C_{11}(\log n)^{1/\rho}}{n} \right) \\ &\le C_{12} \frac{(\log n)^{2/\rho}}{n}. \end{aligned} \tag{1.7}$$

Now we consider $|x| > (4\omega^{-1} \log n)^{1/\rho}$. By Remark 1 following Lemma 2 we have, for such $|x|$,

$$\frac{1}{P(x)} \le |x|.$$

By construction,

$$q(x) = \sum_{k \le n} a_k x^k.$$

Hence, for all large n,

$$\begin{aligned} &\frac{1}{P(x)}\left| \frac{1}{f(|x|)} - \frac{1}{q(x)} \right| \\ &\quad \le |x| \left(\frac{1}{f(|x|)} + \frac{1}{\sum_{k \le n} a_k x^k} \right) \\ &\quad \le (4\omega^{-1} \log n)^{1/\rho} \left\{ \frac{1}{f[(4\omega^{-1} \log n)^{1/\rho}]} + \frac{1}{\sum_{k \le n} a_k (4\omega^{-1} \log n)^{k/\rho}} \right\} \\ &\quad \le \left(\frac{4}{\omega} \log n \right)^{1/\rho} 3\{ f[(4\omega^{-1} \log n)^{1/\rho}] \}^{-1}. \end{aligned} \tag{1.8}$$

Since

$$\sum_{k\leqslant n} a_k(4\omega^{-1}\log n)^{k/\rho} = f[(4\omega^{-1}\log n)^{1/\rho}] - \sum_{k\geqslant n+1} a_k(4\omega^{-1}\log n)^{k/\rho},$$

and

$$\sum_{k\geqslant n+1} a_k(4\omega^{-1}\log n)^{k/\rho} \leqslant \sum_{k\geqslant n+1}\left(\frac{\rho e\tau(1+\epsilon)\,4\omega^{-1}\log n}{k}\right)^{k/\rho} \leqslant \frac{1}{n^{1/2}},$$

we have

$$\sum_{k\leqslant n} a_k(4\omega^{-1}\log n)^{k/\rho} \geqslant f[(4\omega^{-1}\log n)^{1/\rho}] - \frac{1}{n^{1/2}} \geqslant 2^{-1} f[(4\omega^{-1}\log n)^{1/\rho}].$$

By using the definition of lower type, we get

$$f[(4\omega^{-1}\log n)^{1/\rho}] \geqslant \exp(4(1-\epsilon)\log n) > n^3. \tag{1.9}$$

Equation (1.1) follows from (1.5)–(1.9). If $f(z)$ is even, then by using $S_n(x)$, the nth partial sum of $f(x)$, instead of $q(x)$, in (1.7), we get for $0 \leqslant |x| \leqslant (4\omega^{-1}\log n)^{1/\rho}$, by Lemmas 2 and 6, for some $\delta > 1$,

$$\frac{1}{P(x)}\left|\frac{1}{f(x)} - \frac{1}{S_n(x)}\right| \leqslant \left(|x| + \frac{C_{14}(\log n)^{1/\rho}}{n}\right)\delta^{-n} < n^{-3}. \tag{1.10}$$

For $|x| \geqslant (4\omega^{-1}\log n)^{1/\rho}$, by using Remark 2 following Lemma 2 we get, for all large n,

$$\begin{aligned}\frac{1}{P(x)}\left|\frac{1}{f(x)} - \frac{1}{S_n(x)}\right| &\leqslant \frac{|x|}{f(x)} + \frac{|x|}{S_n(x)}\\ &\leqslant \frac{(4\omega^{-1}\log n)^{1/\rho}}{f((4\omega^{-1}\log n)^{1/\rho})} + \frac{(4\omega^{-1}\log n)^{1/\rho}}{S_n((4\omega^{-1}\log n)^{1/\rho})}\\ &\leqslant \frac{3(4\omega^{-1}\log n)^{1/\rho}}{f((4\omega^{-1}\log n)^{1/\rho})},\end{aligned} \tag{1.11}$$

since as earlier

$$2S_n((4\omega^{-1}\log n)^{1/\rho}) \geqslant f((4\omega^{-1}\log n)^{1/\rho}).$$

The Remark after Theorem 1 follows from (1.5), (1.6), (1.9), (1.10), and (1.11).

THEOREM 2. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geqslant 0$ $(k \geqslant 0)$, be an entire function of order ρ $(0 < \rho < \infty)$ and type τ $(0 < \tau < \infty)$. Then for every polynomial $P(x)$ of large degree n, we have,*

$$\left\|\frac{x^{1/2}}{f(x^{1/2})} - \frac{1}{P(x)}\right\|_{L_\infty[0,\infty)} \geqslant \left(\frac{\log n}{2\tau}\right)^{1/\rho} \frac{(9n)^{-1}}{f[(\log n/2\tau)^{2/\rho} n^{-2}]}. \tag{2.1}$$

Proof. Assume the conclusion is false. Then for infinitely many n,

$$\left\| \frac{x^{1/2}}{f(x^{1/2})} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \leq \left(\frac{\log n}{2\tau}\right)^{1/\rho} \frac{(9n)^{-1}}{f[(\log n/2\tau)^{2/\rho} n^{-2}]}. \tag{2.2}$$

Set $\beta_n = ((\log n)/2\tau)^{1/\rho}$, $n = 1, 2, \ldots$. From (2.2) we get, for

$$x \in [\beta_n^2 n^{-2}, \beta_n^2], \qquad \frac{1}{P(x)} \geq \frac{x^{1/2}}{f(x^{1/2})} - \left(\frac{\log n}{2\tau}\right)^{1/\rho} \frac{(9n)^{-1}}{\psi_n}$$
$$\geq \frac{\beta_n n^{-1}}{\psi_n} - \frac{\beta_n n^{-1}}{9\psi_n} = \frac{8}{9} \beta_n n^{-1} \psi_n^{-1},$$

where

$$\psi_n = f(\beta_n n^{-1}).$$

Hence

$$\max_{[\beta_n^2 n^{-2}, \beta_n]} |P(x)| \leq (9/8)\, n\psi_n \beta_n^{-1}. \tag{2.3}$$

Now by applying Lemma 3 to (2.3), we get

$$P(0) \leq \frac{9}{8} n\psi_n \beta_n^{-1} T_n \left(\frac{n^2+1}{n^2-1}\right) \leq 9n\psi_n\beta_n^{-1}. \tag{2.4}$$

On the other hand, we have by (2.2),

$$P(0)^{-1} \leq \beta_n (9n\psi_n)^{-1}. \tag{2.5}$$

Equations (2.4) and (2.5) clearly contradict (2.2).

THEOREM 3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geq 0$ $(k \geq 1)$, be an even entire function of order ρ $(0 < \rho < \infty)$ type τ and lower type ω $(0 < \omega \leq \tau < \infty)$. Then there exists a constant $C_{15} > 0$ and a sequence of rational functions $r_{2n}(x)$ of degree at most $2n$ for which, for all large n,*

$$\| (|x|/f(x)) - r_{2n}(x) \|_{L_\infty(-\infty,\infty)} \leq e^{-C_{15} n^{1/2}}. \tag{3.1}$$

Proof. $|x|/f(|x|)$, $S_n(x)$, and $Q_n^*(x) = n^{1/2\rho} Q_n(xn^{-1/2\rho})$ are even functions. Set $r_{2n}(x) = Q_n^*(x)/S_n(x)$. Then

$$\left| \frac{|x|}{f(x)} - r_{2n}(x) \right| = \left| \frac{x}{f(x)} - \frac{Q_n^*(x)}{f(x)} + \frac{Q_n^*(x)}{f(x)} - \frac{Q_n^*(x)}{S_n(x)} \right|$$
$$\leq \left| \frac{x}{f(x)} - \frac{Q_n^*(x)}{f(x)} \right| + |Q_n^*(x)| \left| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right|. \tag{3.2}$$

Each of the above functions being even, we consider only $[0, \infty)$.

By Lemma 1, for all large n,

$$\left|\frac{x}{f(x)}-\frac{Q_n^*(x)}{f(x)}\right|_{L_\infty[0,n^{1/2\rho}]} \leqslant a_0^{-1}\,|x-Q_n^*(x)|_{L_\infty[0,n^{1/2\rho}]} \leqslant e^{-C_{16}n^{1/2}}. \tag{3.3}$$

On the other hand, for $x \leqslant n^{1/2\rho}$, by the definition of lower type,

$$\begin{aligned}\left|\frac{x}{f(x)}-\frac{Q_n^*(x)}{f(x)}\right| &\leqslant \frac{1}{f(x)}\,|x-Q_n^*(x)|\\ &\leqslant e^{-x^\rho\omega(1-\epsilon)}x\left|\frac{P_n^*(-xn^{-1/2\rho})}{P_n^*(xn^{-1/2\rho})+P_n^*(-xn^{-1/2\rho})}\right|\\ &< e^{-C_{17}n^{1/2}},\end{aligned} \tag{3.4}$$

for n such that

$$P_n^*(-xn^{-1/2\rho}) \geqslant 0.$$

Similarly, we show for $[0, n^{1/2\rho}]$,

$$|Q_n^*(x)|\left|\frac{1}{f(x)}-\frac{1}{S_n(x)}\right| \leqslant (|x| + C_{18}n^{1/2\rho}e^{-C_{19}n^{1/2}})\,e^{-C_{20}n^{1/2}} \leqslant e^{-C_{21}n^{1/2}}. \tag{3.5}$$

On the other hand, for $x^{2\rho} \geqslant n > n_0$,

$$\begin{aligned}|Q_n^*(x)|\left|\frac{1}{f(x)}-\frac{1}{S_n(x)}\right| &\leqslant \left|\frac{Q_n^*(x)}{f(x)}\right| + \left|\frac{Q_n^*(x)}{S_n(x)}\right| \leqslant |x|\left(\frac{1}{f(x)}+\frac{1}{S_n(x)}\right)\\ &\leqslant n^{1/2\rho}\left(\frac{1}{f(n^{1/2\rho})}+\frac{1}{S_n(n^{1/2\rho})}\right) \leqslant \frac{3n^{1/2\rho}}{f(n^{1/2\rho})}\\ &\leqslant e^{-C_{22}n^{1/2}}.\end{aligned} \tag{3.6}$$

As earlier, it is easy to check that $2S_n(n^{1/2\rho}) \geqslant f(n^{1/2\rho})$, and also that $|Q_n^*(x)| \leqslant |x|$. Hence (3.1) follows from (3.3)–(3.6).

THEOREM 4. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, $a_k \geqslant a_{k+1}$ $(k \geqslant 1)$, *be a noneven entire function of order* ρ $(0 < \rho < \infty)$, *type* τ, *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Then there exists a rational function* $r_{2n}^*(x)$ *of degree at most* $2n$ *for which, for all large* n,

$$\left\|\frac{|x|}{f(|x|)} - r_{2n}^*(x)\right\|_{L_\infty(-\infty,\infty)} \leqslant e^{-C_{24}n^{1/2}}, \tag{4.1}$$

where $r_{2n}^*(x) = r_n(x)\,Q_n^*(x)$.

Proof. By Lemma 5,

$$\left\| \frac{1}{f(|x|)} - r_n(x) \right\|_{L_\infty(-\infty,\infty)} \le e^{-C_{25} n^{1/2}}. \tag{4.2}$$

Now write

$$\left| \frac{|x|}{f(|x|)} - r_{2n}(x) \right| \le \left| \frac{|x|}{f(|x|)} - \frac{Q_n^*(x)}{f(|x|)} + \frac{Q_n^*(x)}{f(|x|)} - Q_n^*(x)\, r_n(x) \right|$$
$$\le \frac{1}{f(|x|)} \big| |x| - Q_n^*(x) \big| + |Q_n^*(x)| \left| \frac{1}{f(|x|)} - r_n(x) \right|. \tag{4.3}$$

As earlier. for $0 \le x \le n^{1/2p}$, we get

$$\frac{1}{f(|x|)} \big| |x| - Q_n^*(x) \big| \le C_{26} n^{1/2p} e^{-C_{27} n^{1/2}} \le e^{-C_{28} n^{1/2}}. \tag{4.4}$$

On the other hand, for $|x| \ge n^{1/2p}$,

$$\frac{1}{f(|x|)} \big| |x| - Q_n^*(x) \big|$$
$$\le e^{-x^{p}(1-\epsilon)} x \left| \frac{P_n^*(xn^{-1/2p}) - P_n^*(-xn^{-1/2p})}{P_n^*(xn^{-1/2p}) + P_n^*(-xn^{-1/2p})} \right| \le e^{-C_{29} n^{1/2}}, \tag{4.5}$$

if $P_n^*(-xn^{-1/2p}) \ge 0$. Similarly, for $0 \le |x| \le n^{1/2p}$,

$$|Q_n^*(x)| \left| \frac{1}{f(|x|)} - r_n(x) \right| \le (|x| + n^{1/2p} e^{-C_{30} n^{1/2}}) e^{-C_{31} n^{1/2}}. \tag{4.6}$$

On the other hand, for $|x| \ge n^{1/2p}$,

$$|Q_n^*(x)| \left| \frac{1}{f(|x|)} - r_n(x) \right| \le \frac{|x|}{f(|x|)} + \frac{|x|}{\sum_{k \le n} a_{2k} x^{2k}}, \tag{4.7}$$

since from the construction of $r_n(x)$,

$$r_n(x) \le \left(\sum_{k \le n} a_{2k} x^{2k} \right)^{-1}.$$

By our assumption on the coefficients, we have

$$2 \sum_{k \le n} a_{2k} x^{2k} \ge \sum_{j \le 2n} a_j x^j \tag{4.8}$$

As before, we can show

$$2 \sum_{l \leqslant n} a_l n^{l/2\rho} \geqslant f(n^{1/2\rho}). \tag{4.9}$$

From (4.7), (4.8), and (4.9), we get for $|x| > n^{1/2\rho}$,

$$|Q^*(x)| \left| \frac{1}{f(|x|)} - r_n(x) \right| \leqslant \frac{5n^{1/2\rho}}{e^{n^{1/2}\omega(1-\epsilon)}} \leqslant e^{-C_{22}n^{1/2}}. \tag{4.10}$$

Equation (4.1) follows from (4.4), (4.5), (4.6), and (4.10).

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